PHYC 581: High Energy Astrophysics
Fall 2014

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Thin Disk Theory (Cont'd):

The power exerted on the ring at radius R, due to the net viscous torque, is:

$$P = -\Omega \frac{d\Gamma_{out}}{dR} dR$$

It can be written as:

$$P = -\left[\frac{d}{dR}(\Gamma_{out}\Omega) - \Gamma_{out}\Omega' \right] dR$$

The first term inside the brackets represents a global transfer of rotational energy through the disk. The second term corresponds to an actual local dissipation that produces heat. As long as the dissipated energy, which is coming from the conversion of gravitational potential energy, is radiated efficiently, the disk will remain geometrically thin. Otherwise, the thin-disk approximation is no longer valid. We will discuss this in more detail later on.

Assuming that the dissipated energy is radiated from the surface of the disk (i.e., disk is optically thick), the rate for dissipation per unit surface area is:

$$D(R) = \frac{T_{out}\,\Omega'}{4\pi R} = -\frac{1}{2}\nu\Sigma\,(R\Omega')^2$$

After using the mass conservation and angular momentum equations, we find:

$$R\Sigma v_R R^2 \Omega = -\frac{1}{2\pi} T_{out} + \text{Const.} \Rightarrow -\nu\Sigma\Omega' = \Sigma(-v_R)\Omega + \frac{\text{Const.}}{R^3}$$

The constant can be found by considering near the surface of the compact object (a white dwarf or a neutron star, black holes do not have a hard surface in which case we consider the last stable orbit). Ω eventually merges with rotation at the surface, implying that $\Omega' \to 0$ and $\Omega \to \Omega_K(R_*)$ (R_* being the object's radius). This leads to:

$$\text{Const.} = -\frac{\dot{M}}{2\pi}\left(GMR_*\right)^{\frac{1}{2}}$$

Finally, we arrive in an expression for ν as a function of $\dot{M}$:

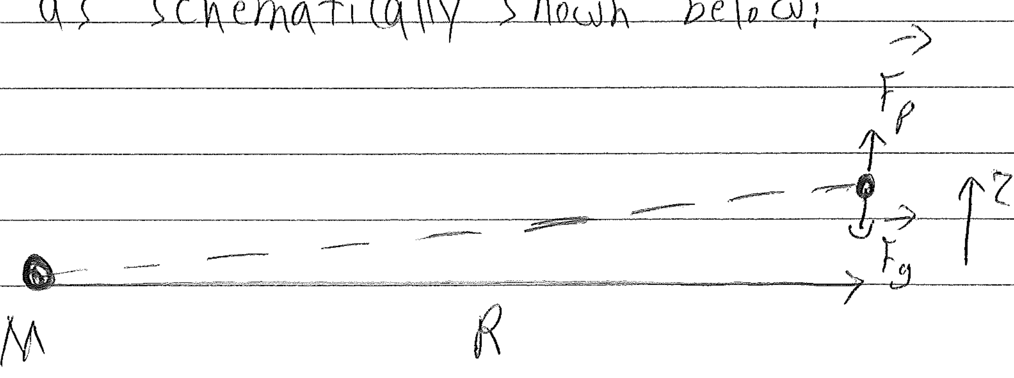
$$\nu \Sigma = \frac{\dot{M}}{3\pi}\left[1-\left(\frac{R_*}{R}\right)^{1/2}\right]$$

Thus:

$$D_{(R)} = \frac{3GM\dot{M}}{8\pi R^3}\left[1-\left(\frac{R_*}{R}\right)^{1/2}\right]$$

This expression for dissipation rate as a function of physical parameters is useful when calculating the spectrum of radiation from a thin disk.

To verify the thinness of the disk, let us consider the vertical structure of the disk. We may assume that the gas is in local hydrostatic equilibrium in the vertical direction as schematically shown below:



Here $\vec{F}_g$ and $\vec{F}_p$ denote the gravitational force and the force from pressure gradient per unit volume respectively, where:

$$\vec{F}_g \approx -\frac{GM\rho}{R^2}\sin\theta\,\hat{z} = -\frac{GM\rho z}{R^3}\,\hat{z}$$

$$\vec{F}_p = -\frac{\partial P}{\partial z}\,\hat{z}$$

The condition for hydrostatic equilibrium states that:

$$\vec{F}_g + \vec{F}_p = 0 \Rightarrow \frac{1}{\rho}\frac{\partial P}{\partial z} = -\frac{GMz}{R^3}$$

If the gas is isothermal in the vertical direction (which is the case if temperature gradient in the vertical direction is sufficiently small, as usually happens on the disk), we have:

$$\frac{\partial P}{\partial z} = \frac{R_g}{\nu}\,T_{(R)}\,\frac{\partial \rho}{\partial z}$$

Here we have used the equation of state of an isothermal ideal gas, where R_g is the gas constant and ν is the mean molecular weight per particle ($\nu=1$ for a gas

Consisting of Hydrogen).

The above equation can be easily solved, resulting in:

$$\rho_{(R,z)} = \rho_c(R) \exp\left(\frac{-z^2}{2\zeta^2}\right) \qquad \zeta \equiv \left(\frac{R^3 R_g T}{G M \mu}\right)^{\frac{1}{2}}$$

Here $\rho_c(R)$ is the density at the center of the disk $(z=0)$, and ζ can be interpretted as half of the disk thickness $\frac{H}{2}$.

As an example consider the structure of the disk when the compact object has a mass $M_\odot$. In this case, as we will see later, $T \sim 10^4 - 10^5$ K, which results in:

$$\frac{\zeta}{R} \sim 10^{-7} \left(\frac{R}{1\,m}\right)^{\frac{1}{2}}$$

For $R \sim 10$ km (as in a neutron star), we have $\zeta \sim 10^{-5} R$. For $R \sim 10^5$ km (as one may have around a white dwarf), we find $\zeta \sim 10^{-3} R$. Therefore the thin-disk geometry is well justified in both of the cases,